

Central configurations, Morse and fixed point indices

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Abstract

We compute the fixed point index of non-degenerate central configurations for the n -body problem in the euclidean space of dimension d , relating it to the Morse index of the gravitational potential function \bar{U} induced on the manifold of all maximal $O(d)$ -orbits. In order to do so, we analyze the geometry of maximal orbit type manifolds, and compute Morse indices with respect to the mass-metric bilinear form on configuration spaces.

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1 Introduction: central configurations as critical points

Let $E = \mathbb{R}^d$ be the d -dimensional euclidean space, for $d \geq 1$. Fix an integer $n \geq 2$. The *configuration space* of n (colored) points in E is the set of all n -tuples of distinct points in E , and denoted by $\mathbb{F}_n(E)$:

$$\mathbb{F}_n(E) = \{\mathbf{q} \in E^n : i \neq j \implies \mathbf{q}_i \neq \mathbf{q}_j\} = E^n \setminus \Delta,$$

where if $\mathbf{q} \in E^n$, its n components are denoted by \mathbf{q}_j , $j = 1, \dots, n$; points in $\mathbb{F}_n(E)$ are termed *configurations* of n points in E ; its complement in E^n is the set of *collisions*

$$\begin{aligned} \Delta &= \{\mathbf{q} \in E^n : \exists(i, j), i \neq j : \mathbf{q}_i = \mathbf{q}_j\} \\ &= \bigcup_{1 \leq i < j \leq n} \{\mathbf{q} \in E^n : \mathbf{q}_i = \mathbf{q}_j\}. \end{aligned}$$

For $j = 1, \dots, n$ let $m_j > 0$ be fixed parameters (that can be interpreted as the mass of the j -th particle in E), under the normalization condition

$$\sum_{j=1}^n m_j = 1 .$$

If \mathbf{v}, \mathbf{w} are vectors in (the tangent space of) E^n , then let

$$\langle \mathbf{v}, \mathbf{w} \rangle_M = \sum_{j=1}^n m_j \mathbf{v}_j \cdot \mathbf{w}_j$$

denote the mass scalar product of \mathbf{v} and \mathbf{w} , where $\mathbf{v}_j \cdot \mathbf{w}_j$ is the standard euclidean scalar product (in E) of the j -th components of \mathbf{v} and \mathbf{w} . The unit sphere in $\mathbb{F}_n(E)$ is termed the *inertia ellipsoid* and denoted by

$$\mathbb{S} = \mathbb{S}_n(E) = \{\mathbf{q} \in \mathbb{F}_n(E) : \|\mathbf{q}\|_M^2 = 1\} .$$

It is equal to the unit sphere/ellipsoid in E^n , with collisions removed, $\mathbb{S}_n(E) = S_n(E) \setminus \Delta$. The unit sphere/ellipsoid in E^n is denoted by $S_n(E) = \{\mathbf{q} \in E^n : \|\mathbf{q}\|_M^2 = 1\}$. To simplify notation, if possible we will use the short forms \mathbb{S} and S instead of $\mathbb{S}_n(E)$ and $S_n(E)$.

The *potential function* $U: \mathbb{F}_n(E) \rightarrow \mathbb{R}$ is simply defined as

$$\sum_{1 \leq i < j \leq n} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|^\alpha},$$

given a fixed parameter $\alpha > 0$. For $\alpha = 1$, U is the gravitational potential. It is invariant under the full group of isometries of E , acting diagonally on $\mathbb{F}_n(E)$.

Let $D = \nabla$ denote the covariant derivative (which is the Levi-Civita connection with respect to the mass-metric) in $\mathbb{F}_n(E)$, which is again the standard derivative. If $F: \mathbb{F}_n(E) \rightarrow E$ is a smooth function, then $DF = dF$ is the differential of F , which is a section of the cotangent bundle $T^*\mathbb{F}_n(E)$ defined as $DF[\mathbf{v}] = D_{\mathbf{v}}F$ for each vector field \mathbf{v} on $\mathbb{F}_n(E)$. If \mathbf{v} and \mathbf{w} are two vector fields on $\mathbb{F}_n(E)$, then $D_{\mathbf{v}}\mathbf{w}$ is the (Euclidean and covariant) derivative of \mathbf{w} in the direction of \mathbf{v} .

Let ∇^S denote the covariant derivative (Levi-Civita connection) on S , induced by the mass-metric of $\mathbb{F}_n(E)$ restricted to S , i.e. the restriction to S of the Riemannian structure of $\mathbb{F}_n(E)$. If \mathbf{v} and \mathbf{w} are two vector fields defined in a neighborhood of S , then the covariant derivative $\nabla_{\mathbf{v}}^S \mathbf{w}$ is equal, at $x \in S$, to the orthogonal projection of $D_{\mathbf{v}}\mathbf{w}$, projected orthogonally to the tangent space $T_x S$ (cf. proposition 3.1 at page 11 of [7], or proposition

1.2 at page 371 of [8]). The same holds with $\mathbb{S} \subset S$ instead of S . If Π denote the projection $T\mathbb{F}_n(E) \mapsto T\mathbb{S}$, then $\nabla_{\mathbf{v}}^S \mathbf{w} = \Pi D_{\mathbf{v}} \mathbf{w}$.

If $F: \mathbb{F}_n(E) \rightarrow \mathbb{R}$ is a smooth function, and $f = F|_{\mathbb{S}}$ is its restriction to \mathbb{S} , then $\nabla^S f = df$ is the restriction of dF to the tangent bundle $T\mathbb{S}$. Let $\text{grad}(f) = df^\sharp$ and $\text{grad}(F) = dF^\sharp$ denote the gradients of f and F respectively (i.e., the images of the differentials under the musical isomorphisms induced by the mass-metric). For each $x \in \mathbb{S}$, $df^\sharp(x) \in T_x \mathbb{S}$ and $dF^\sharp(x) \in T_x \mathbb{F}_n(E)$ satisfy the equations

$$\langle df^\sharp, \mathbf{v} \rangle_M = df[\mathbf{v}] = \langle dF^\sharp, \mathbf{v} \rangle_M = dF[\mathbf{v}]$$

for any $\mathbf{v} \in T_x \mathbb{S}$, and hence $\text{grad}(f) = df^\sharp$ is the projection of $\text{grad}(F) = dF^\sharp$ on the tangent space $T_x \mathbb{S}$. A *critical point* of f is a point $x \in \mathbb{S}$ such that $df = 0 \iff \text{grad}(f) = 0$, which is equivalent to say that $\text{grad}(F)$ is orthogonal to $T_x \mathbb{S}$.

The *Hessian* of the function f , at a critical point x of f in \mathbb{S} , is (cf. page 343 of [8]) equal to the bilinear form $\text{Hess}(f)[\mathbf{v}, \mathbf{w}]$, defined on the tangent space $T_x \mathbb{S}$ as

$$\text{Hess}(f)[\mathbf{v}, \mathbf{w}](x) = \nabla_{\mathbf{v}}^S \nabla_{\mathbf{w}}^S f - \nabla_{\nabla_{\mathbf{v}}^S \mathbf{w}}^S f(x) = (\nabla_{\mathbf{v}}^S \nabla_{\mathbf{w}}^S f)(x)$$

where \mathbf{v} and \mathbf{w} are two vector fields defined in a neighborhood of x .

The Hessian of F is simply the symmetric matrix of all the second derivatives $D^2 F$:

$$\begin{aligned} \text{Hess}(F)[\mathbf{v}, \mathbf{w}](x) &= (D_{\mathbf{v}} D_{\mathbf{w}} F)(x) = D^2 F(x)[\mathbf{v}, \mathbf{w}] \\ &= \sum_{\substack{i=1, \dots, n \\ \beta=1, \dots, d}} \sum_{\substack{j=1, \dots, n \\ \gamma=1, \dots, d}} \frac{\partial^2 F}{\partial \mathbf{q}_{i\beta} \partial \mathbf{q}_{j\gamma}} \mathbf{v}_{i\beta} \mathbf{w}_{j\gamma} \end{aligned}$$

where $\mathbf{q}_{i\beta}$, $\mathbf{v}_{i\beta}$ and $\mathbf{w}_{j\gamma}$ are the d cartesian components in E (\mathbb{R}^d as the tangent space of E) of \mathbf{q}_i , \mathbf{v}_i and \mathbf{w}_j respectively.

Using the mass-metric, if \mathbf{N} denotes the unit vector field normal to $T_x \mathbb{S}$ in $T_x \mathbb{F}_n(E)$, the projection of $\nabla_{\mathbf{v}}^S \mathbf{u}$ of any vector field \mathbf{u} on $T_x \mathbb{S}$ is

$$\nabla_{\mathbf{v}}^S \mathbf{u} = D_{\mathbf{v}} \mathbf{u} - \langle D_{\mathbf{v}} \mathbf{u}, \mathbf{N} \rangle_M \mathbf{N},$$

and

$$df^\sharp = dF^\sharp - \langle dF^\sharp, \mathbf{N} \rangle_M \mathbf{N}.$$

The Hessian can be written also as (cf. page 344 of [8]) $\text{Hess}(f)[\mathbf{v}, \mathbf{w}](x) = \langle \nabla_{\mathbf{v}}^S df^\sharp, \mathbf{w} \rangle_M$ and $\text{Hess}(F)[\mathbf{v}, \mathbf{w}](x) = \langle D_{\mathbf{v}} dF^\sharp, \mathbf{w} \rangle_M$. It follows therefore that

$$\begin{aligned} \text{Hess}(f)[\mathbf{v}, \mathbf{w}](x) &= \langle \nabla_{\mathbf{v}}^S df^\sharp, \mathbf{w} \rangle_M \\ &= \langle \nabla_{\mathbf{v}}^S (dF^\sharp - \langle dF^\sharp, \mathbf{N} \rangle_M \mathbf{N}), \mathbf{w} \rangle_M \\ &= \langle \nabla_{\mathbf{v}}^S (dF^\sharp), \mathbf{w} \rangle_M - \langle \nabla_{\mathbf{v}}^S (\langle dF^\sharp, \mathbf{N} \rangle_M \mathbf{N}), \mathbf{w} \rangle_M. \end{aligned}$$

Because of the product rule for each function φ and each vector field \mathbf{u}

$$\begin{aligned}\nabla_{\mathbf{v}}^S(\varphi\mathbf{u}) &= \varphi\nabla_{\mathbf{v}}^S\mathbf{u} + (d\varphi[\mathbf{v}])\mathbf{u} \\ \implies \nabla_{\mathbf{v}}^S(\langle dF^\sharp, \mathbf{N} \rangle_M \mathbf{N}) &= \langle dF^\sharp, \mathbf{N} \rangle_M \nabla_{\mathbf{v}}^S \mathbf{N} + d(\langle dF^\sharp, \mathbf{N} \rangle_M) [\mathbf{v}] \mathbf{N}\end{aligned}$$

which implies that

$$\langle \nabla_{\mathbf{v}}^S(\langle dF^\sharp, \mathbf{N} \rangle_M \mathbf{N}), \mathbf{w} \rangle_M = \langle dF^\sharp, \mathbf{N} \rangle_M \langle \nabla_{\mathbf{v}}^S \mathbf{N}, \mathbf{w} \rangle_M$$

since \mathbf{N} is orthogonal to \mathbf{w} . The same argument can be applied to show that for any vector field \mathbf{u} (not necessarily tangent to \mathbb{S})

$$\langle \nabla_{\mathbf{v}}^S \mathbf{u}, \mathbf{w} \rangle_M = \langle D_{\mathbf{v}} \mathbf{u}, \mathbf{w} \rangle_M,$$

and therefore that, evaluated at the critical point x ,

$$\begin{aligned}\text{Hess}(f)[\mathbf{v}, \mathbf{w}] &= \langle D_{\mathbf{v}}(dF^\sharp), \mathbf{w} \rangle_M - \langle dF^\sharp, \mathbf{N} \rangle_M \langle D_{\mathbf{v}} \mathbf{N}, \mathbf{w} \rangle_M \\ &= D^2 F[\mathbf{v}, \mathbf{w}] - \langle dF^\sharp, \mathbf{N} \rangle_M \langle D_{\mathbf{v}} \mathbf{N}, \mathbf{w} \rangle_M\end{aligned}$$

The inertia ellipsoid S is defined by the equation $\|\mathbf{q}\|_M^2 = 1$, or equivalently $h(\mathbf{q}) = \frac{1}{2}$ where $h(\mathbf{q}) = \frac{1}{2}\|\mathbf{q}\|_M^2$. The normal unit vector \mathbf{N} is equal to $dh^\sharp = \mathbf{q}$, and thus

$$\begin{aligned}\text{Hess}(f)[\mathbf{v}, \mathbf{w}] &= D^2 F[\mathbf{v}, \mathbf{w}] - \langle dF^\sharp, \mathbf{q} \rangle_M \langle D_{\mathbf{v}} \mathbf{q}, \mathbf{w} \rangle_M \\ &= D^2 F[\mathbf{v}, \mathbf{w}] - \langle dF^\sharp, \mathbf{q} \rangle_M \langle \mathbf{v}, \mathbf{w} \rangle_M\end{aligned}$$

If $F = U$, then U is homogeneous of degree $-\alpha$, and therefore $\langle dU^\sharp, \mathbf{q} \rangle_M = dU(\mathbf{q})[\mathbf{q}] = -\alpha U(\mathbf{q})$. The following equation follows, at any critical point x of the restriction of U to \mathbb{S} .

$$(1.1) \quad \text{Hess}(U|_{\mathbb{S}})[\mathbf{v}, \mathbf{w}] = D^2 U(x)[\mathbf{v}, \mathbf{w}] + \alpha U(x) \langle \mathbf{v}, \mathbf{w} \rangle_M$$

A *central configuration* is a configuration $\mathbf{q} \in \mathbb{F}_n(E)$ with the property that there exists a multiplier $\lambda \in \mathbb{R}$ such that

$$(1.2) \quad dU^\sharp(\mathbf{q}) = \lambda \mathbf{q},$$

where dU^\sharp is the gradient in E^n of the potential function U , with respect to the mass-metric. Equation (1.2) implies that $\lambda = -\alpha \frac{U(\mathbf{q})}{\|\mathbf{q}\|_M^2}$ (for more on central configurations see e.g. [17] (§369–§382bis at pp. 284–306), [15], [10], [12], [18], [1], [6], [2], [11], [5]). An equivalent definition for a normalized (i.e. $\mathbf{q} \in \mathbb{S}$) central configuration is the following:

(1.3) $\mathbf{q} \in \mathbb{S}_n(E)$ is a central configuration if and only if it is a critical point for the restriction $U|_{\mathbb{S}}$ of the potential function to $\mathbb{S} = \mathbb{S}_n(E)$.

Let $c: E^n \rightarrow E^n$ be the isometry defined as $c(\mathbf{q}) = \mathbf{q}'$, with

$$(1.4) \quad \mathbf{q}'_j = \mathbf{q}_j - 2\mathbf{q}_0$$

for each $j = 1, \dots, n$, and with $\mathbf{q}_0 = \sum_{j=1}^n m_j \mathbf{q}_j$. It is an isometry, since $\|\mathbf{q}'\|_M^2 = \sum_{j=1}^n m_j |\mathbf{q}_j - 2\mathbf{q}_0|^2 = \sum_{j=1}^n m_j (|\mathbf{q}_j|^2 + 4|\mathbf{q}_0|^2 - 4\mathbf{q}_j \cdot \mathbf{q}_0) = \sum_{j=1}^n m_j |\mathbf{q}_j|^2 + 4(\sum_{j=1}^n m_j) |\mathbf{q}_0|^2 - 4|\mathbf{q}_0|^2 = \|\mathbf{q}\|_M^2$. It is the orthogonal reflection around the space of all configurations with center of mass \mathbf{q}_0 equal to zero: $c\mathbf{q} = \mathbf{q} \iff \mathbf{q}_0 = 0$. It is easy to see that if \mathbf{q} is a central configuration then $c\mathbf{q} = \mathbf{q}$, and hence \mathbf{q} has center of mass \mathbf{q}_0 in 0. Let Y be defined as $Y = \{\mathbf{q} \in E^n : \mathbf{q}_0 = \mathbf{0}\}$, and $\mathbb{S}^c = \mathbb{S} \cap Y$, $S^c = S \cap Y$. In other words, elements of \mathbb{S}^c are normalized configurations with center of mass in 0. Since the potential function is invariant up to translations, $U(c\mathbf{q}) = U(\mathbf{q})$, and any critical point of the restriction $U|_{\mathbb{S}^c}$ is a critical point of $U|_{\mathbb{S}}$ (for example, by Palais Principle of Symmetric Criticality [13]). Thus it is equivalent to define central configurations as critical points of $U|_{\mathbb{S}^c}$ or as critical points of $U|_{\mathbb{S}}$.

2 Fixed points, $SO(d)$ -orbits and projective configuration spaces

Following [3, 4], consider the function $F: \mathbb{S}_n(E) \rightarrow S_n(E)$ defined as

$$(2.1) \quad F(\mathbf{q}) = -\frac{dU^\sharp(\mathbf{q})}{\|dU^\sharp(\mathbf{q})\|_M}$$

where dU^\sharp is the gradient of U , with respect to the mass-metric.

First, consider the isometry c defined above in (1.4). Since $F(c\mathbf{q}) = cF(\mathbf{q})$, $F(\mathbb{S}^c) \subset S^c$. Moreover, as the image of F is in S^c , if F^c denotes the restriction $F^c: \mathbb{S}^c \rightarrow S^c$,

$$(2.2) \quad \text{Fix}(F^c) = \text{Fix}(F),$$

and the fixed point indexes are exactly the same.

Let $O(d)$ be the special orthogonal group, acting diagonally on E^n , and $SO(d)$ the special orthogonal subgroup. The inertia ellipsoid \mathbb{S} , S and Y are $O(d)$ -invariant in E^n , and so are \mathbb{S}^c and S^c . Let $\pi: S \rightarrow S/G$ denote the quotient map onto the space of G -orbits, for $G = SO(d)$ or $G = -O(d)$.

Since U is a G -invariant function, F is a G -equivariant map, and hence it induces a map on the quotient spaces:

$$(2.3) \quad \begin{array}{ccc} \mathbb{S} & \xrightarrow{F} & S \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{S}/G & \xrightarrow{f} & S/G \end{array}$$

A fixed point of F is a normalized configuration \mathbf{q} such that $F(\mathbf{q}) = \mathbf{q}$. A fixed point of f is a conjugacy class $[\mathbf{q}]$ of configurations such that $f([\mathbf{q}]) = [\mathbf{q}]$, i.e. it is a conjugacy class $[\mathbf{q}]$ such that $F(\mathbf{q}) = g\mathbf{q}$ for some $g \in G$. It follows from Theorem (2.5) of [4] that if $G = SO(d)$, then $F(\mathbf{q}) = g\mathbf{q} \iff F(\mathbf{q}) = \mathbf{q}$, or equivalently that

$$(2.4) \quad G = SO(d) \implies \pi(\text{Fix}(F)) = \text{Fix}(f),$$

and hence also that $\pi(\text{Fix}(F^c)) = \text{Fix}(f^c)$.

(2.5) *Remark.* Elements in \mathbb{S}/G are called projective configurations: for $d = 2$ and $G = SO(2)$, S/G is the $(n - 1)$ -dimensional complex projective space $\mathbb{P}^{n-1}(\mathbb{C})$, and S^c is a hyperplane in it, hence a $(n - 2)$ -dimensional complex projective space $\mathbb{P}^{n-2}(\mathbb{C})$. For $n = 3$, it is the Riemann sphere. Projective configurations are projective classes of elements $[\mathbf{q}_1 : \mathbf{q}_2 : \mathbf{q}_3]$ in $\mathbb{P}^1(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C})$ such that $m_1\mathbf{q}_1 + m_2\mathbf{q}_2 + m_3\mathbf{q}_3 = 0$, $\mathbf{q}_j \in \mathbb{C}$, and $\mathbf{q}_1 \neq \mathbf{q}_2$, $\mathbf{q}_1 \neq \mathbf{q}_3$, $\mathbf{q}_2 \neq \mathbf{q}_3$.

For $d = 1$, projective configurations are equivalence classes under the action of the orthogonal group $G = O(1) = \mathbb{Z}_2$.

The following Corollary of (2.4) shows that the difference is minor.

(2.6) Corollary. *If $\mathbf{q} \in \mathbb{S}$ is a central configuration such that $F(\mathbf{q}) = g\mathbf{q}$, with $g \in O(d)$ (acting diagonally on E^n), then $g = 1$.*

Proof. Let $E' = E \oplus \mathbb{R}$ be the euclidean space of dimension $d+1$, and $E \subset E'$ one of its d -dimensional subspaces. If $\mathbf{q} \in \mathbb{S} \subset \mathbb{F}_n(E)$, then $\mathbf{q} \in \mathbb{S} \subset \mathbb{F}_n(E) \subset \mathbb{F}_n(E')$, and there exists $g' \in SO(d+1)$ such that $g'E = E$ and the restriction of g' to E is equal to g : it follows that $F(\mathbf{q}) = g'\mathbf{q}$, in $\mathbb{F}_n(E')$, and therefore $g' = 1$, from which it follows that $g = 1$. *q.e.d.*

Homological calculations on configurations spaces for the sake of central configurations have been done by Palmore [14], Pacella [12] and McCord [9]. We can arrange all the spaces inertia ellipsoids and the corresponding

projective quotients as in diagram (2.7).

$$\begin{array}{ccccccc}
 & \mathbb{S}_n^c(\mathbb{R}) & \xrightarrow{\iota_1} & \mathbb{S}_n^c(\mathbb{R}^2) & \xrightarrow{\iota_2} & \mathbb{S}_n^c(\mathbb{R}^3) & \xrightarrow{\iota_3} \dots \\
 & \downarrow & & \downarrow & & \downarrow & \\
 (2.7) \quad & \mathbb{S}_n^c(\mathbb{R})/SO(1) & \xrightarrow{\bar{\iota}_1} & \mathbb{S}_n^c(\mathbb{R}^2)/SO(2) & \xrightarrow{\bar{\iota}_2} & \mathbb{S}_n^c(\mathbb{R}^3)/SO(3) & \xrightarrow{\bar{\iota}_3} \dots \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \mathbb{S}_n^c(\mathbb{R})/O(1) & \xrightarrow{\bar{\iota}_1} & \mathbb{S}_n^c(\mathbb{R}^2)/O(2) & \xrightarrow{\bar{\iota}_2} & \mathbb{S}_n^c(\mathbb{R}^3)/O(3) & \xrightarrow{\bar{\iota}_3} \dots
 \end{array}$$

For each d , $\mathbb{S}_n^c(\mathbb{R}^d)$ is a deformation retract of $\mathbb{F}_n^c(\mathbb{R}^d)$, which in turn is a deformation retraction of $\mathbb{F}_n(\mathbb{R}^d)$ (where $\mathbb{F}_n^c(E)$ denotes the space of all configurations with center of mass in 0). The Poincaré polynomial for the cohomology of the configuration space $\mathbb{F}_n(\mathbb{R}^d)$ is equal to

$$P(t) = \prod_{k=1}^{n-1} (1 + kt^{d-1}),$$

as shown e.g. in Theorem 3.2 of [16] (see also Proposition 2.11.2 of [11]).

Now, note that in the sequence of projections

$$\mathbb{S}_n^c(\mathbb{R}^d) \rightarrow \mathbb{S}_n^c(\mathbb{R}^d)/SO(d) \rightarrow \mathbb{S}_n^c(\mathbb{R}^d)/O(d)$$

the second map corresponds to the projection given by the action of the quotient group $\mathbb{Z}_2 = O(d)/SO(d)$ on the quotient space $\mathbb{S}_n^c(\mathbb{R}^d)/SO(d)$ ($SO(d)$ is normal in $O(d)$). For $d \geq 2$, let h be the orthogonal reflection of \mathbb{R}^d around $\mathbb{R}^{d-1} \subset \mathbb{R}^d$: its coset $hSO(d)$ is the generator of $O(d)/SO(d)$, and hence the image $\text{Im}(\bar{\iota}_{d-1})$ in $\mathbb{S}_n^c(\mathbb{R}^d)/SO(d)$ is fixed by $O(d)/SO(d)$. Actually, it is equal to the fixed point subset of $O(d)/SO(d)$ in $\mathbb{S}_n^c(\mathbb{R}^d)/SO(d)$. Outside the image of $\bar{\iota}_{d-1}$, therefore the \mathbb{Z}_2 action is free: let $\mathbb{M}_n(\mathbb{R}^d)$ denote the manifold

$$\mathbb{M}_n(\mathbb{R}^d) = (\mathbb{S}_n^c(\mathbb{R}^d)/SO(d) \setminus \text{Im}(\bar{\iota}_{d-1})) / \mathbb{Z}_2 = \mathbb{S}_n^c(\mathbb{R}^d)/O(d) \setminus \text{Im}(\bar{\iota}_{d-1}),$$

where the last equality holds since $\bar{\iota}_{d-1}$ factors through $\mathbb{S}_n^c(\mathbb{R}^{d-1})$.

The next proposition follows from the dimension of $SO(d)$ and the previous remarks.

(2.9) *The subspace of all points in $\mathbb{S}_n^c(\mathbb{R}^d)/O(d)$ with maximal orbit type is the open subspace $\mathbb{M}_n(\mathbb{R}^d)$ defined in (2.8), and it is a manifold of dimension*

$$\dim \mathbb{M}_n(\mathbb{R}^d) = d(n-1) - 1 - d(d-1)/2$$

For $d = 1$, it is the projective space $\mathbb{P}^{n-2}(\mathbb{R})$ minus collisions. For $d = 2$, it is a $(2n - 4)$ dimensional manifold (where $\mathbb{P}^{n-2}(\mathbb{C})$ minus collinear and minus collisions is its double cover).

(2.10) $\mathbb{S}_n^c(\mathbb{R}^2)/SO(2)$ has the same homotopy type of $\mathbb{F}_{n-2}(\mathbb{R}^2 \setminus \{p, q\})$, where p, q are two arbitrary distinct points of \mathbb{R}^2 .

Proof. It is Lemma 4.1 of [9].

q.e.d.

It follows that the Poincaré polynomial (where β_j are Betti numbers) of $\mathbb{S}_n^c(\mathbb{R}^2)/SO(2)$ is

$$(2.11) \quad p(t) = \prod_{k=2}^{n-1} (1 + kt) = \sum_{j=0}^{n-2} \beta_j t^j.$$

(see also Proposition 2.11.3 of [11]). McCord in [9] proved also that

$$\dim H^k(\mathbb{M}_n(\mathbb{R}^2)) = \begin{cases} \sum_{j=0}^k \beta_j & \text{if } k \leq n-3 \\ 0 & \text{otherwise,} \end{cases}$$

while Pacella in (2.4) of [12] computed the $SO(3)$ -equivariant homology (using Borel homology) Poincaré series of $\mathbb{S}_n^c(\mathbb{R}^3) \sim \mathbb{F}_n(\mathbb{R}^3)$ as

$$P^{SO(3)}(t) = \frac{\prod_{k=2}^{n-1} (1 + kt^2)}{1 - t^2}$$

(2.12) Remark. The projective quotient $\mathbb{S}_n^c(\mathbb{R}^2)/SO(2)$ is a manifold (it is the projective space $\mathbb{P}^{n-2}(\mathbb{C})$ with collisions removed). It contains $\mathbb{S}_n^c(\mathbb{R})/O(1)$ as a submanifold (the collinear configurations). For $d \geq 3$ the isotropy groups of the action start being non-trivial, and the filtration of subspaces of constant orbits type in $\mathbb{S}_n^c(\mathbb{R}^d)/SO(d)$ is given by the horizontal arrows \bar{t}_j in diagram (2.7).

3 Fixed points and Morse indices

Let $\mathbf{q} \in \mathbb{S}_n^c(\mathbb{R}^d)$ a central configuration, and hence a fixed point of the map F defined above in (2.1), such that its $O(d)$ -orbits lies in the maximal orbit type submanifold $\mathbb{M}_n(\mathbb{R}^d) \subset \mathbb{S}^c(\mathbb{R}^d)/O(d)$.

(3.1) If $DF: T_{\mathbf{q}}\mathbb{S} \rightarrow T_{\mathbf{q}}\mathbb{S}$ denotes the differential of F at the central configuration \mathbf{q} , then for any $\mathbf{v}, \mathbf{w} \in T_{\mathbf{q}}\mathbb{S}$ the following equation holds:

$$D^2U(\mathbf{q})[\mathbf{v}, \mathbf{w}] = -\alpha U(\mathbf{q}) \langle DF[\mathbf{v}], \mathbf{w} \rangle_M$$

Proof. As we have seen in the introduction, $\langle D_v dU^\sharp, \mathbf{w} \rangle_M = D^2 U[\mathbf{v}, \mathbf{w}]$, and if \mathbf{q} is a normalized central configuration then by (1.2) $dU^\sharp(\mathbf{q}) = \lambda \mathbf{q}$ with $\lambda = -\alpha \frac{U(\mathbf{q})}{\|\mathbf{q}\|_M^2} = -\alpha U(\mathbf{q})$. It follows that $\langle dU^\sharp, \mathbf{w} \rangle_M = 0$, being \mathbf{w} tangent to \mathbb{S} , and $\|dU^\sharp\|_M = -\lambda = \alpha U(\mathbf{q})$. Also,

$$\begin{aligned} \langle DF[\mathbf{v}], \mathbf{w} \rangle_M &= \langle D_v \left(-\frac{dU^\sharp}{\|dU^\sharp\|_M} \right), \mathbf{w} \rangle_M \\ &= -\left\langle \left(\frac{D_v dU^\sharp}{\|dU^\sharp\|_M} \right), \mathbf{w} \right\rangle_M - \left\langle D_v \left(\frac{1}{\|dU^\sharp\|_M} \right) dU^\sharp, \mathbf{w} \right\rangle_M \\ &= -\frac{1}{\|dU^\sharp\|_M} \langle D_v dU^\sharp, \mathbf{w} \rangle_M - 0 \\ &= -\frac{1}{\alpha U(\mathbf{q})} D^2 U(\mathbf{q})[\mathbf{v}, \mathbf{w}]. \end{aligned}$$

q.e.d.

Combining (3.1) with equation (1.1) the following corollary holds.

(3.2) Corollary. *If \mathbf{q} is as above, then for each $\mathbf{v}, \mathbf{w} \in T_{\mathbf{q}}\mathbb{S}$*

$$\text{Hess}(U|_{\mathbb{S}})[\mathbf{v}, \mathbf{w}] = \alpha U(\mathbf{q}) (\langle \mathbf{v}, \mathbf{w} \rangle_M - \langle DF[\mathbf{v}], \mathbf{w} \rangle_M)$$

Finally, consider again the group $O(d)$ acting on $\mathbb{S}_n^c(\mathbb{R}^d)$. Let F and \mathbf{q} be the map and the central configuration defined above. Recall that $f: \mathbb{S}/O(d) \rightarrow S/O(d)$ denotes the map defined on the quotient. Let $[\mathbf{q}] \in \mathbb{M}_n(\mathbb{R}^d)/\mathbb{S}/O(d)$ denote the projective class (i.e. the $O(d)$ -orbit of \mathbf{q}) of \mathbf{q} , which is a fixed point of f , and is a critical point of the map $\bar{U}: \mathbb{M}_n(\mathbb{R}^d) \rightarrow \mathbb{R}$ induced on \mathbb{M}_n by U , defined simply as $\bar{U}([x]) = U(x)$ for each $x \in \mathbb{S}_n^c(\mathbb{R}^d)$.

(3.3) Theorem. *The point $[\mathbf{q}]$ is a non-degenerate critical point of \bar{U} if and only if it is a non-degenerate fixed point of f . If $\text{ind}([\mathbf{q}], f)$ denotes the fixed point index of $[\mathbf{q}]$ for f , and $\mu([\mathbf{q}])$ the Morse index of $[\mathbf{q}]$, then the following equation holds:*

$$\text{ind}([\mathbf{q}], f) = (-1)^{\mu([\mathbf{q}])}.$$

Proof. The point $[\mathbf{q}]$ is a non-degenerate critical point if and only if the dimension of the kernel of the Hessian $\text{Hess}(U|_{\mathbb{S}})(\mathbf{q})$ is equal to the dimension of $SO(d)$, i.e. $d(d-1)/2$. By (3.2), the kernel is equal to the eigenspace of $DF(\mathbf{q})$ corresponding to the eigenvalue 1, which has dimension $d(d-1)/2$ if and only if the fixed point $[\mathbf{q}]$ is non-degenerate. Now, if this holds then the index $\text{ind}([\mathbf{q}], f)$ is equal to the number $(-1)^e$, where e is the number

of negative eigenvalues $1 - f'$, which is the same as the number of negative eigenvalues of $1 - F'$. Again by (3.2) and since $U > 0$, e is equal to the number of negative eigenvalues of $\text{Hess}(U|_{\mathbb{S}})$, which is by definition the Morse index $\mu([\mathbf{q}])$. *q.e.d.*

(3.4) *Remark.* Unfortunately, a former version of this statement had a wrong formula for $\text{ind}(\mathbf{q})$. In fact, in (3.5) of [4] one should put $\epsilon = 0$, and not $\epsilon = d(n - 1) - 1 - d(d - 1)/2 = \dim \mathbb{M}_n(\mathbb{R}^d)$. The error occurred because I used the wrong sign of U in (3.1) ($V = -U$ instead of U).

(3.5) Example. For $d = 1$ and any n , all critical points are local minima of U , and hence $\mu = 0$, and fixed points have index 1. The map induced on the quotient can be regularized on binary collisions (see [4, 3]), hence the map on the quotient can be extended to a self-map $f: \mathbb{P}^1(\mathbb{R}) \rightarrow \mathbb{P}^1(\mathbb{R})$ with three fixed points of index 1. Therefore the Lefschetz number of f is 3, and f has degree -2 .

For $d = 2$ and $n = 3$, the three Euler configurations have $\mu = 1$, while the two Lagrange points have $\mu = 1$, hence the map f induced on the quotient $\mathbb{P}^1(\mathbb{C})$ (again, by regularizing the binary collisions) has Lefschetz number equal to $L(f) = 2 - 3 = -1$. Therefore the degree of f is equal to -2 .

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